

# Enhanced quantization on the circle

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## Abstract

We apply the quantization scheme introduced in [arXiv:1204.2870] to a particle on a circle. We find that the quantum action functional restricted to appropriate coherent states can be expressed as the classical action plus  $\hbar$ -corrections. This result extends the examples presented in the cited paper.

Pacs numbers: 03.65.-w

Key words: Quantization, coherent states, periodic coordinates.

March 7, 2013

# 1 Introduction

Conventional canonical quantization promotes a classical momentum  $p$  into a Hermitian operator  $P$  and a classical position  $q$  into a Hermitian operator  $Q$  which obey  $[Q, P] = i\hbar$ . Other classical quantities such as the Hamiltonian find a quantum counterpart as  $H(p, q) \rightarrow \mathcal{H} = H(P, Q)$  plus possible corrections of  $O(\hbar)$ . This prescription works very well for many systems but it also has its limitations.

Enhanced quantization [1][2] offers a new interpretation of the very process of quantization that encompasses the usual canonical story and offers additional features as well. Provided both  $P$  and  $Q$  are self-adjoint operators (a stronger condition than Hermiticity) we can generate unitary operators acting on a fiducial state  $|\eta\rangle$  to yield

$$|p, q\rangle = e^{-\frac{i}{\hbar}qP} e^{\frac{i}{\hbar}pQ} |\eta\rangle \quad (1)$$

as a set of coherent states which spans the Hilbert space  $\mathfrak{H}$ . If the quantum action functional

$$A_Q = \int_0^T \langle \psi(t) | [i\hbar\partial_t - \mathcal{H}] | \psi(t) \rangle dt \quad (2)$$

is used in a variational principle, it leads to the Schrödinger equation  $[i\hbar\partial_t - \mathcal{H}]|\psi(t)\rangle = 0$  as an equation of motion. However, arbitrary variations of  $|\psi(t)\rangle$  for a microscopic system are not accessible to a macroscopic observer who can only change the velocity or position of the microscopic system leading to the fact that the only state she/he could make are those represented by  $|p(t), q(t)\rangle$ . The restricted quantum action functional becomes

$$\begin{aligned} A_{Q(R)} &= \int_0^T \langle p(t), q(t) | [i\hbar\partial_t - \mathcal{H}] | p(t), q(t) \rangle dt \\ &= \int_0^T [p(t)\dot{q}(t) - H(p(t), q(t))] dt \end{aligned} \quad (3)$$

which, because  $\hbar > 0$  still, may be called an enhanced classical action functional whereas the usual classical action is given by

$$A_C = \int_0^T [p(t)\dot{q}(t) - H_c(p(t), q(t))] dt, \quad (4)$$

where  $H_c(p(t), q(t)) = \lim_{\hbar \rightarrow 0} H(p(t), q(t))$ . In this view, then, classical theory is a subset of quantum theory, and they both co-exist just like they do in the real world where  $\hbar > 0$  [1].

Other two-dimensional continuous sheets of unit vectors may also correspond to a classical canonical system, and in [1], two other sets of coherent states were shown to have this property. In this paper, we consider the enhanced quantization of a classical particle moving on a circle of finite radius. This system leads to yet another set of coherent states which serves to unite the classical and quantum theories for such a system. This is our main result.

Quantum mechanics on a nontrivial topological space has been discussed in many contexts and still attracts attention (see [3, 4, 5] and more references therein) and coherent states on the circle have been also discussed in [6, 7, 8, 9, 10]. Nevertheless, none of these contributions addresses the issue of the relationship between classical and quantum actions as presented here. This paper provides another instance for which the rationale introduced in [1] yields an interesting quantum/classical connection.

## 2 Enhanced quantization on the circle

Consider a particle on the circle  $S^1$ . The position space can be parametrized by  $\theta \in [-\pi, \pi)$ . We consider, at the quantum level, a set of quantum operators  $Q$  and  $P$ , associated with position and momentum, satisfying the commutation relation:

$$[Q, P] = i\hbar. \quad (5)$$

The spectrum of the operator  $Q$  is bounded in  $[-\pi, \pi)$  and, like  $\theta$ , is periodic. To start the program of [1], the operators  $P$  and  $Q$  have to be self-adjoint such that the operators  $e^{-\frac{i}{\hbar}qP}$  and  $e^{-\frac{i}{\hbar}pQ}$  keep their ordinary and useful unitary feature.

**Self-adjoint extension of  $P$  on the circle.** Self-adjoint extensions are well known on the space of square integrable functions on any finite interval  $[A, B]$ , with vanishing boundary conditions (see for instance [11]). In a streamlined analysis, we review the properties of  $P = -i\hbar\partial_\theta$  as an operator acting on  $L^2([-\pi, \pi), d\theta)$ .

Let us investigate the domain  $D(P)$  of  $P$  for  $P$  being symmetric, i.e.,  $P^\dagger = P$  on  $D(P)$ . Consider the inner product for any two functions  $\psi, \varphi, \psi', \varphi' \in L^2([-\pi, \pi), d\theta)$  (with as yet unspecified boundary values) given by

$$(\psi, P\varphi) = (-i\hbar) \int_{-\pi}^{\pi} \psi^*(\theta) \varphi'(\theta) d\theta = (-i\hbar) \psi^* \varphi \Big|_{-\pi}^{\pi} + (P\psi, \varphi), \quad (6)$$

so that for  $(\psi, P\varphi) = (P\psi, \varphi)$  to hold on  $D(P)$ , one requires

$$\psi^*(\pi)\varphi(\pi) - \psi^*(-\pi)\varphi(-\pi) = 0. \quad (7)$$

This condition is satisfied if we adopt  $\varphi(\pm\pi) = 0$  and make no restriction on  $\psi$ . In this case  $D(P) = \{\varphi; \varphi, \varphi' \in L^2([-\pi, \pi), d\theta); \varphi(\pi) = \varphi(-\pi) = 0\}$ . However, the domain of  $P^\dagger$ ,  $D(P^\dagger) = \{\varphi; \varphi, \varphi' \in L^2([-\pi, \pi), d\theta)\} \supset D(P)$ .

Defining the self-adjoint extension of  $P$  is a procedure aimed at rendering  $D(P^\dagger) = D(P)$  by enlarging  $D(P)$  and restricting  $D(P^\dagger)$  so they coincide [11]. In particular, the condition (7) can also be fulfilled by imposing the boundary condition,  $\varphi(\pi) = e^{2\pi i\alpha}\varphi(-\pi)$ , for a given  $\alpha \in [0, 1)$ , and thus we can enlarge the domain of  $P$  and reduce the domain of  $P^\dagger$  so that

$$\tilde{D}(P_\alpha) = \left\{ \varphi; \varphi, \varphi' \in L^2([-\pi, \pi), d\theta); \varphi(\pi) = e^{2\pi i\alpha}\varphi(-\pi) \right\} = \tilde{D}(P_\alpha^\dagger). \quad (8)$$

As noticed in [1], having defined self-adjoint extensions for  $P_\alpha$  and  $Q$  (which is trivial here), we can define unitary operators by exponentiating these generators.

**Coherent states on the circle.** Let us now pursue the quantum program associated with (5) henceforth denoting  $P$  by  $P_\alpha$ , where  $\alpha \in [0, 1)$  labels the different inequivalent representations of the momentum operator. We will use units such that  $Q$  is dimensionless and so the dimension of  $P$  is that of  $\hbar$ . We define eigenvectors  $|\theta\rangle$  for the operator  $Q$ , satisfying  $\langle\theta|\theta'\rangle = \delta_{S^1}(\theta - \theta')$ , where  $\delta_{S^1}$  should be understood as periodic on  $S^1$ , as well as eigenvectors  $|n, \alpha\rangle$  of  $P_\alpha$ , obeying  $\langle n, \alpha|m, \alpha\rangle = \delta_{n,m}$ , such that

$$Q|\theta\rangle = \theta|\theta\rangle, \quad P_\alpha|n, \alpha\rangle = p_{n,\alpha}|n, \alpha\rangle, \quad \langle\theta|P_\alpha|n, \alpha\rangle = (-i\hbar)\partial_\theta\langle\theta|n, \alpha\rangle = p_{n,\alpha}\langle\theta|n, \alpha\rangle. \quad (9)$$

It is well known that the spectrum of  $P_\alpha$  on the circle is such that  $p_{n,\alpha} = \hbar(n + \alpha)$ ,  $(n, \alpha) \in \mathbb{Z} \times [0, 1)$  (more features introduced by the topology of the manifold and self-adjoint properties of  $P$  can be found in [3][4][5]). As a realization of the functions  $\langle \theta | n, \alpha \rangle$ , we find that the normalized wave functions are given by

$$\langle \theta | n, \alpha \rangle = \frac{1}{\sqrt{2\pi}} e^{i(n+\alpha)\theta}. \quad (10)$$

The self-adjoint operators  $P_\alpha$  and  $Q$  yield the unitary operators  $e^{-\frac{i}{\hbar}qP_\alpha}$  and  $e^{-\frac{i}{\hbar}pQ}$ , where  $(q, p) \in S^1 \times \mathbb{R}$ . From these operators, a set of states is defined by

$$|p, q\rangle = e^{-\frac{i}{\hbar}qP_\alpha} e^{\frac{i}{\hbar}pQ} |\eta_\alpha\rangle, \quad (11)$$

where  $|\eta_\alpha\rangle$  is called the fiducial state. We verify that the set  $\{|p, q\rangle\}$  satisfies

- (i) A normalization condition:  $\langle p, q | p, q \rangle = \langle \eta_\alpha | \eta_\alpha \rangle = 1$ , since  $\langle \eta_\alpha | \eta_\alpha \rangle$  is normalized.
- (ii) A resolution of unity:

$$\int_{\mathbb{R} \times S^1} |p, q\rangle \langle p, q| \frac{dp dq}{2\pi \hbar} = I_{\mathfrak{H}}. \quad (12)$$

Indeed, it can be shown that, for all  $\theta, \theta' \in [\pi, \pi)$ ,

$$\int_{\mathbb{R} \times S^1} \langle \theta | p, q \rangle \langle p, q | \theta' \rangle \frac{dp dq}{2\pi \hbar} = \delta_{S^1}(\theta - \theta') \int_{S^1} \langle \theta - q | \eta_\alpha \rangle \langle \eta_\alpha | \theta - q \rangle dq = \langle \theta | \theta' \rangle \langle \eta_\alpha | \eta_\alpha \rangle, \quad (13)$$

where we used the fact that  $\int_{\theta-\pi}^{\theta+\pi} dq |q\rangle \langle q| = \int_{-\pi}^{\pi} dq |q\rangle \langle q|$  because of periodicity. Thus (12) is recovered for a normalized  $|\eta_\alpha\rangle$ .

The set of states  $\{|p, q\rangle\}$  forms an overcomplete family of normalized states. Henceforth, these states will be called coherent states.

We can now discuss the dynamics associated with such states by introducing a general quantum Hamiltonian of the form  $\mathcal{H}(P, e^{iQ}, e^{-iQ})$ . It has been argued that quantum and classical mechanics should co-exist (see for instance [12]–[16]). In the present situation and using the coherent states, we establish a link between the quantum and the classical actions.

Consider the restricted quantum action associated with  $|\psi(t)\rangle \rightarrow |p(t), q(t)\rangle$  defined above, which leads to

$$A_{Q(R)} = \int_0^T \langle p(t), q(t) | \left[ i\hbar \partial_t - \mathcal{H} \right] | p(t), q(t) \rangle dt. \quad (14)$$

As explained earlier, within this action functional a macroscopic observer can vary, not the entire Hilbert space of states, but only the coherent-state subspace, when studying a microscopic system. We choose a class of fiducial vectors satisfying

$$\langle \eta_\alpha | Q | \eta_\alpha \rangle = 0 \quad \text{and} \quad \langle \eta_\alpha | P_\alpha | \eta_\alpha \rangle = \hbar \alpha \quad (15)$$

so that  $|\eta_\alpha\rangle$  is in the domain of the self-adjoint operators  $Q$  and  $P_\alpha$ .

In the ordinary canonical situation, the choice of the fiducial vector  $|\eta\rangle$  as the ground state of an harmonic oscillator makes it an extremal weight vector:  $(Q + iP)|\eta\rangle = 0$ , and the latter relation yields  $\langle\eta|Q|\eta\rangle = 0$  and  $\langle\eta|P|\eta\rangle = 0$ . Hence, (15) can be considered an analog of this condition modified slightly due to the topology of the configuration space.

A straightforward calculation using (15) yields

$$\langle p(t), q(t) | [i\hbar\partial_t] | p(t), q(t) \rangle = (\hbar\alpha + p)\dot{q}. \quad (16)$$

Furthermore, we have

$$H_\alpha(p(t), q(t)) := \langle p(t), q(t) | \mathcal{H}(P_\alpha, e^{iQ}, e^{-iQ}) | p(t), q(t) \rangle = \langle \eta_\alpha | \mathcal{H}(P_\alpha + p, e^{i(Q+q)}, e^{-i(Q+q)}) | \eta_\alpha \rangle. \quad (17)$$

Hence, the restricted quantum action reads

$$A_{Q(R)} = \int_0^T \left\{ [p\dot{q} - H_\alpha(p(t), q(t))] + \hbar\alpha\dot{q} \right\} dt, \quad (18)$$

where the term  $\tilde{A}_C = \int_0^T [p\dot{q} - H_\alpha(p(t), q(t))] dt$ , as in the ordinary situation [1], can be related to a classical action  $A_C$  up to  $\hbar$  corrections since

$$H_\alpha(p(t), q(t)) = H_{c,\alpha}(p, q) + O(\hbar; p, q). \quad (19)$$

In the last equation,  $H_{c,\alpha}(p, q)$  is viewed as the usual classical Hamiltonian. Interestingly, we notice that the quantum parameter  $\alpha$  induces a surface term  $\hbar\alpha\dot{q}$  in  $A_{Q(R)}$  which makes no influence on the enhanced, or the classical, equations of motion whatsoever. At the end, we can write

$$A_{Q(R)} = A_C + O(\hbar). \quad (20)$$

Let us discuss, in more detail, a general case which can be evaluated completely. We choose the quantum Hamiltonian as

$$\mathcal{H}(P_\alpha, e^{iQ}, e^{-iQ}) = P_\alpha^2 + V(e^{iQ}, e^{-iQ}), \quad V(e^{iQ}, e^{-iQ}) = a_0 + \sum_{n=1}^m [a_n \cos nQ + b_n \sin nQ], \quad (21)$$

with mass units chosen so that  $1/2\mu = 1$  and where  $m \in \mathbb{N}$ . Next, we choose a particular fiducial vector such that, for  $0 < b < 1$ ,

$$\eta_\alpha(\theta) := \langle \theta | \eta_\alpha \rangle = N e^{i\alpha\theta} [1 + b \cos \theta]^{r/\hbar}, \quad N = \left[ 2\pi [1 - b]^{2r/\hbar} {}_2F_1 \left( \frac{1}{2}, -2\frac{r}{\hbar}; 1; -\frac{2b}{1-b} \right) \right]^{-1/2}, \quad (22)$$

with  $r/\hbar > m$  an integer,  ${}_2F_1$  denotes the ordinary hypergeometric function [17] and  $N$  is a normalization factor fixed such that  $\langle \eta_\alpha | \eta_\alpha \rangle = 1$ . Note that  $|\eta_\alpha(\theta)|$  is even and periodic and that (15) is satisfied. For large  $r/\hbar \gg 1$ , the following approximation is valid:

$$|\eta_\alpha(\theta)|^2 = |\langle \theta | \eta_\alpha \rangle|^2 = N^2 e^{\frac{2r}{\hbar} [\ln(1+b \cos \theta)]} \leq N^2 e^{\frac{2r}{\hbar} [-\frac{b\theta^2}{2(1+b)} - \ln(1+b)]} = \tilde{N}^2 e^{-\frac{r}{\hbar} \frac{b\theta^2}{(1+b)}}; \quad (23)$$

hence  $|\theta| \lesssim \sqrt{\hbar/r}$  which is small. One can therefore consider  $\eta_\alpha(\theta)$  as cutting-off large  $\theta$ -values. Evaluating the diagonal coherent state matrix elements of  $\mathcal{H}$ , and setting  $\alpha' = \hbar\alpha$ , leads to

$$\begin{aligned} \langle p(t), q(t) | \mathcal{H}(P_\alpha, e^{iQ}, e^{-iQ}) | p(t), q(t) \rangle &= \langle \eta_\alpha | (P_\alpha + p)^2 + V(e^{i(Q+q)}, e^{-i(Q+q)}) | \eta_\alpha \rangle \\ &= (\alpha' + p)^2 - \alpha'^2 + \langle \eta_\alpha | P^2 | \eta_\alpha \rangle + a_0 + \sum_{n=1}^m \left[ a_n \cos nq + b_n \sin nq \right] + O(\hbar) \\ &= (p + \alpha')^2 + V(e^{iq}, e^{-iq}) + O(\hbar), \end{aligned} \quad (24)$$

where  $\langle \eta_\alpha | P_\alpha^2 | \eta_\alpha \rangle$  and  $\alpha'^2$  are constants included in  $O(\hbar)$ . Thus, we can infer that

$$A_{Q(R)} = \int_0^T \left\{ (p + \alpha')\dot{q} - \left[ (p + \alpha')^2 + V(e^{iq}, e^{-iq}) \right] + O(\hbar) \right\} dt. \quad (25)$$

Therefore, up to constants and a canonical shift in momentum ( $p \rightarrow p + \alpha'$ ),

$$A_{Q(R)} = \int_0^T \left\{ p\dot{q} - \left[ p^2 + V(e^{iq}, e^{-iq}) \right] + O(\hbar) \right\} dt = A_C + O(\hbar). \quad (26)$$

Canonical transformations are well defined in the present setting. These transformations involve a change of variables  $(p, q) \rightarrow (\tilde{p}, \tilde{q})$  such that the symplectic structure at the classical level is preserved leading to  $\{\tilde{q}, \tilde{p}\} = 1 = \{q, p\}$  as well as  $pdq = \tilde{p}d\tilde{q} + d\tilde{G}(\tilde{p}, \tilde{q})$ , where  $\tilde{G}$  is the generator of such a transformation. Coherent states in the transformed coordinates, namely  $|\tilde{p}, \tilde{q}\rangle$ , are chosen to be the same as before the change of variables. Thus, it is clear that the restricted quantum action computed with respect to  $|\tilde{p}, \tilde{q}\rangle \equiv |p(\tilde{p}, \tilde{q}), q(\tilde{p}, \tilde{q})\rangle = |p, q\rangle$  yields a classical action  $\tilde{A}_C$  corresponding to  $A_C$  up to a surface term given by  $\tilde{G}(\tilde{p}, \tilde{q})$ . No change of the operators is involved.

**Coherent state induced geometry on phase space.** The geometry of the coherent states [14] can be investigated as well using the Fubini-Study metric element,  $\tilde{c}_\alpha ds_\alpha^2 = ||d|p, q\rangle||^2 - |\langle p, q | d|p, q \rangle|^2$ , given  $\tilde{c}_\alpha = c_\alpha/\hbar^2$ . We have  $d|p, q\rangle = -(i/\hbar)e^{-\frac{i}{\hbar}qP_\alpha}e^{\frac{i}{\hbar}pQ}\left[dq(P_\alpha + p) - dpQ\right]|\eta_\alpha\rangle$ , and therefore

$$\begin{aligned} ||d|p, q\rangle||^2 &= \frac{1}{\hbar^2} \left[ dq^2(D'_\alpha(r) + 2p\alpha' + p^2) + dp^2 D_\alpha(r) \right], \\ |\langle p, q | d|p, q \rangle|^2 &= \frac{1}{\hbar^2} |\langle \eta_\alpha | \left[ dq(P_\alpha + p) - dpQ \right] | \eta_\alpha \rangle|^2 = \frac{1}{\hbar^2} (p + \alpha')^2 dq^2, \end{aligned} \quad (27)$$

where  $D_\alpha(r) := \langle \eta_\alpha | Q^2 | \eta_\alpha \rangle$  and  $D'_\alpha(r) := \langle \eta_\alpha | P^2 | \eta_\alpha \rangle$  are both constants. Finally, the metric on the subspace of coherent states can be written as

$$c_\alpha ds_\alpha^2 = (D'_\alpha(r) - \alpha'^2) dq^2 + D_\alpha(r) dp^2, \quad ds_\alpha^2 = A_\alpha dq^2 + dp^2 = dq_\alpha^2 + dp^2, \quad (28)$$

where we have set  $q_\alpha = \sqrt{|A_\alpha|}q$ ,  $c_\alpha = D_\alpha(r)$  and  $A_\alpha = (D'_\alpha(r) - \alpha'^2)/D_\alpha(r)$ . The metric  $ds_\alpha^2$  describes a flat geometry with a cylindrical topology. If desired, this metric can be imposed on the classical phase space as well.

## Acknowledgements

JRK thanks the Perimeter Institute, Waterloo, Canada, for its hospitality. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

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